B. Math. I Year 2001-2002
I Semester Final Exam

Date:19-11-2001 **Analysis I** **Marks: 50**

Part A (10 marks)

In what follows $f : R \to R$ is continuous. Decide if the following statements are true for all continuous functions f. If true, you need not give a proof; if false indicate so by giving a counter example.

1. $f(G)$ is open whenver G is,
2. $f^{-1}(G)$ is open whenever G is,
3. $f^{-1}(F)$ is closed whenever F is,
4. $f(F)$ is closed whenever F is,
5. $f(K)$ is compact whenever K is,
6. $f^{-1}(K)$ is bounded whenever K is,
7. $(f(x_n))$ is Cauchy whenever (x_n) is,
8. $f(E)$ is dense in R whenever E is,
9. $f(E)$ is connected whenever E is,
10. $f^{-1}(E)$ is an open interval whenever E is,

Part B (40 marks)

1. Find all functions $f : R \to R$ which are differentiable at $x = 0$ such that

$$f(x+y) = f(x) + f(y), x, y \in R.$$

[4]

2. If $\sum\limits_{n=1}^{\infty} a_n$ is a convergent series of positive monotonically decreasing terms show that $na_n \to 0$ as $n \to \infty$. [3]

3. Suppose $f : \boldsymbol{R} \to \boldsymbol{R}$ satisfies the condition

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all $x, y \in \boldsymbol{R}, \alpha, \beta \in [0,1], \alpha + \beta = 1$. Show that f is continuous. [4]

4. Show that $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent. Find the sum. [2]

5. If $f \in C[0,1], f(x) \geq 0$ for al $x \in [0,1]$ and if $\int\limits_0^1 f(x)dx = 0$ then show that $f(x) = 0$ for all $x \in [0,1]$. [3] all

6. Given a closed set $E \subset \boldsymbol{R}$ show that there exists a continuous function $g : \boldsymbol{R} \to \boldsymbol{R}$ such that $E = \{x : g(x) = 0\}$. [3]

7. Show that every $f : [0,1] \to [0,1]$ which is continuous has a fixed point: i.e. there is a point $a \in [0,1]$ such that $f(a) = a$. [4]

8. Show that every continuous function $g : \boldsymbol{R} \to \boldsymbol{R}$ which takes open sets into open sets is monotonic. [5]

9. Suppose $f \in C[0,1]$ is such that

$$\int\limits_0^1 f(x)x^n dx = 0, n = 0, 1, 2, \ldots$$

Show that $f(x) = 0$ for all $x \in [0,1]$. [6]

10. Let $\mathcal{P}[0,1]$ is the set of all polynomials restricted to [0,1]. Show that $f(x) = x(1-x)$ is not an interior point of $\mathcal{P}[0,1]$ in $C[0,1]$. What is the interior of $\mathcal{P}[0,1]$? [6]